数理解析研究所講究録 1315 巻 2003 年 58-76

Analysis of quasilinear hyperbolic equations in the space of BV functions

静岡大学・工学部 菊地 光嗣 (Koji Kikuchi)
Faculty of Engineering, Shizuoka University

Abstract. In the case that f is *linear growth* and *quasiconvex* we treat a system of second order quasilinear hyperbolic equations

$$\frac{\partial^2 u^i}{\partial t^2}(t,x) - \sum_{\alpha=1}^{n} \frac{\partial}{\partial x^\alpha}\{f_{p^i_\alpha}(\nabla u(t,x))\} = 0, \quad i = 1,2,\ldots,N \tag{0.1}$$

in a bounded domain $\Omega \subset \boldsymbol{R}^n$ with initial and boundary conditions

$$u(0,x) = u_0(x), \qquad \frac{\partial u}{\partial t}(0,x) = v_0(x), \qquad x \in \Omega, \tag{0.2}$$

$$u(t,x) = 0, \qquad x \in \partial\Omega. \tag{0.3}$$

Approximate solutions to (0.1)–(0.3) are constructed in Rothe's method and it is proved that a subsequence of them converges to a function u and that, if u satisfies the energy conservation law, then it is a weak solution to (0.1)–(0.3) in the space of functions having bounded variation.

1 Introduction

There are several works on the following nonlinear hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2}(t,x) - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\{(1+|\nabla u(t,x)|^2)^{-1/2}\frac{\partial u}{\partial x_j}\} = 0, \qquad x \in \Omega, \tag{1.1}$$

which is in [5, 9, 10] referred to as an equation of motion of vibrating membrane. This equation does not always have a classical solution globally in time; furthermore it is proved in [8] that in the two dimensional case (1.1) does not always have a classical solution globally in time even though the initial data is smooth and small. Thus a time global solution should be found in a weak sense. When a C^2 class function u satisfies (1.1), multiplying u_t to (1.1) and integrating with respect to x, we obtain the energy conservation law

$$\int_\Omega |u_t(t,x)|^2 dx + \int_\Omega \sqrt{1+|\nabla u|^2}dx = \text{const.}$$

The area functional $u \mapsto \int_\Omega \sqrt{1+|\nabla u|^2}dx$ is finite for $u \in W^{1,1}(\Omega)$, and thus this space is expected to be the appropriate function space for weak solutions to (1.1). But it is not reflexive and thus does not guarantee the weak compactness of bounded sets. While, the relaxed functional of the area functional in the $L^1(\Omega)$ norm

$$A(u,\Omega) := \inf\{\liminf_{j\to\infty}\int_\Omega \sqrt{1+|\nabla u_j|^2}dx; \{u_j\} \subset W^{1,1}(\Omega),\ \text{s-}\lim_{j\to\infty} u_j = u \text{ in } L^1(\Omega)\}$$

is finite whenever the distributional derivative Du is an $\boldsymbol{R}^n$ valued finite Radon measure in Ω. Such a function is called a function of bounded variation in Ω, or simply a BV function in Ω (compare to, for example, [1, 3, 7]). The vector space of all BV functions in Ω is denoted by $BV(\Omega)$. It is a Banach space equipped with the norm $\| u \|_{BV} = \| u \|_{L^1(\Omega)} + |Du|(\Omega)$.[1] For a bounded set B in $BV(\Omega)$, there exist a subsequence $\{u_m\} \subset B$ and a function $u \in BV(\Omega)$ such that $u_m \to u$ strongly in $L^1(\Omega)$ and $Du_m \rightharpoonup Du$ in the sense of distributions. Thus $BV(\Omega)$ satisfies a kind of compactness for bounded sets. These facts suggest that equation (1.1) should be treated in the class of BV functions.

In [5, 9, 10] equation (1.1) is investigated in the space of BV functions. All of these works have obtained basically that *a sequence of approximate solutions to* (1.1) *converges to a function* u *in* $L^\infty((0,T); L^2(\Omega) \cap BV(\Omega))$, and that, *if* u *satisfies the energy conservation law, it is a weak solution to* (1.1) *in the space of BV functions*, which is in the sequel referred to as a *BV solution*. In [5] approximate solutions are constructed by Ritz-Galerkin method, while in [9, 10] by Rothe's method. In [5] a further technical assumption is required, while in [9, 10] it is removed. In [5, 9] the boundary condition is not essentially discussed, while in [10] it is discussed. We more comment on the last point. Seemingly the main theorem of [9] asserts that the function u satisfies the boundary condition; however Dirichlet boundary condition is in fact implicitly assumed in the energy conservation law (compare to [10, Section 1]). The approximation method employed in [9, 10] suggests that the most appropriate weak formulation of Dirichlet condition (0.3) is not to suppose the trace vanishes but to replace $A(u, \Omega)$ with $A(u, \overline{\Omega})$, the value of the measure of $\overline{\Omega}$ defined by $A(u, \cdot)$, where u is regarded as the null extension of u to a domain $\tilde{\Omega}$ containing $\overline{\Omega}$ (for details, refer to [10], in Section 2 we briefly review the definition of a BV solution to (1.1)). Remark that this weaker formulation of (0.3) makes the condition of energy conservation law weaker. In [10] it is proved that the same result still holds even if we only suppose this weaker condition.

Rothe's approximation method employed in [9, 10] is a method of semidiscretization in time variable. Hence in this method we should solve elliptic equations with respect to space variables, and the most effective method of solving an elliptic equation in the BV space is a direct variational method; indeed in [9, 10] elliptic equations are solved by minimizing variational functionals. In this respect this method is essentially the same as the method of minimizing movements. The minimizing movement theory is proposed by E. De Giorgi [6] and in terms of this theory the result in [9, 10] can be said, *if a generalized minimizing movement corresponding to* (1.1) *satisfies energy conservation law, then it is a BV solution.*

The purpose of this article is to establish the same result for vectorial cases. In the sequel the set of all N by n matrices with real elements is simply denoted by $\boldsymbol{R}^{nN}$. Let f be a real valued function defined on $\boldsymbol{R}^{nN}$ and suppose that it is asymptotically linear:

(A1) there exist constants m and M such that

$$m|p| \leq f(p) \leq M(1 + |p|). \tag{1.2}$$

In this article we consider system (0.1) of quasilinear hyperbolic equations. Similarly to the scalar case, if we have a classical solution u to (0.1), multiplying u_t to (0.1) and

[1] Given a vector valued Radon measure μ, we write its total variation as $|\mu|$.

integrating with respect to x, we obtain the following energy conservation law

$$\int_\Omega |u_t(t,x)|^2 dx + \int_\Omega f(\nabla u(x))dx = \text{const.}$$

If

(A2) f is quasiconvex, i.e.,

$$\frac{1}{\mathcal{L}^n(D)} \int_D f(p_0 + \nabla\varphi(x))dx \geq f(p)$$

for each bounded domain $D \subset \boldsymbol{R}^n$, for each $p_0 \in \boldsymbol{R}^{nN}$, and for each $\varphi \in [W_0^{1,\infty}(D)]^N$,

the relaxed functional of the functional $u \mapsto \int_\Omega f(\nabla u(x))dx$ in the $[L^1(\Omega)]^N$ norm, which is denoted by J, is finite for $u = (u^1, u^2, \ldots, u^N) \in [BV(\Omega)]^N$ and is expressed as

$$J(u,\Omega) = \int_\Omega f(\nabla u(x))dx + \int_\Omega f_\infty(\frac{dD^s u}{d|D^s u|})d|D^s u|, \tag{1.3}$$

where $Du = D^a u + D^s u$ (absolutely continuous part and singular part with respect to $\mathcal{L}^n$), $D^a u = \mathcal{L}^n \llcorner \nabla u$, and $f_\infty(p)$ is defined as, for $p \in \boldsymbol{R}^n$,

$$f_\infty(p) = \limsup_{\rho\to 0} f(\frac{p}{\rho})\rho \tag{1.4}$$

(see, for example, [1, Theorem 5.47]). However similarly to the scalar case the most appropriate weak formulation of Dirichlet condition (0.3) is to replace $J(u,\Omega)$ with $J(u,\overline{\Omega})$. The functional $J(u,\overline{\Omega})$ is expressed as

$$J(u,\overline{\Omega}) = J(u,\Omega) + \int_{\partial\Omega} f_\infty(\gamma u \times \vec{n})d\mathcal{H}^{n-1}, \tag{1.5}$$

where $\vec{n}$ denotes the inward pointing unit normal to $\partial\Omega$ and $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure.

Naturally several technical assumptions should be required.

(A3) $f \in C^1(\boldsymbol{R}^{nN})$.

(A4) there exists a constant C such that $|f_p(p)| \leq C$

(A5) $\lim_{\rho\to 0} f_p(\frac{p}{\rho}) : p$ exists and this convergence is uniform with respect to p in a compact subset in $\boldsymbol{R}^{nN}$.

Moreover we should require a strictness of quasiconvexity of f. It is presented in Section 4 (assumption (A6)).

In [9, 10] the main theorem is obtained by the use of varifold theory, more precisely, by corresponding each BV function to a varifold based on its graph and the broken part, passing to a limit in the topology of the class of general varifolds, and investigating the structure of the limit varifold. The purpose of this article is to establish the same fact for vectorial cases. However the graph of a vector valued BV function cannot in general correspond to a varifold as in the scalar case. For this reason the varifold theory is not available in vectorial cases and we should give up observations to geometrical structures of the graph. As a result we are forced to define a BV solution in a somewhat weakened sense.

Suppose that $u_0 \in [L^2(\Omega) \cap BV(\Omega)]^N$ and $v_0 \in [L^2(\Omega)]^N$. In this article we employ the following as a definition of a BV solution to (0.1) with (0.2) and (0.3).

Definition 1.1 A function u is said to be a BV solution to (0.1)–(0.3) in $(0,T)\times\Omega$ if and only if

i) $u \in L^\infty((0,T);BV(\Omega)),\quad u_t \in L^2((0,T)\times\Omega)$

ii) $u(0,x) = u_0(x)$

iii) for any $\phi \in C_0^1([0,T)\times\Omega)$,

$$\int_0^T \{-\int_\Omega u_t\phi_t(t,x)dx + \int_\Omega f_p(\nabla u) : \nabla_x\phi(t,x)dx\}dt = \int_\Omega v_0(x)\phi(0,x)dx$$

iv) for any $\psi \in C_0^1([0,T))$,

$$\begin{aligned} &\int_0^T \{-\int_\Omega u_t(\psi'(t)u + \psi(t)u_t)dx + \psi(t)\int_\Omega f_p(\nabla u) : \nabla u dx \\ +\ & \psi(t)\int_\Omega f_\infty(\frac{dD^s u}{d|D^s u|})d|D^s u| + \psi(t)\int_{\partial\Omega} f_\infty(\gamma u \otimes \vec{n})d\mathcal{H}^{n-1}\}dt \\ =\ & \psi(0)\int_\Omega v_0(x)u_0(x)dx. \end{aligned}$$

This definition is possibly too weak. But, at least, for (1.1), (0.2), (0.3) ($N=1$ and $f(p)=\sqrt{1+|p|^2}$) it is equivalent to the definition of a weak solution to $u_{tt} + \partial A(u) \ni 0$. We briefly review the definition of a BV solution to (1.1) in Section 2. In Section 3 our main theorem is presented (Theorem 3.3) and give a proof except for the convergence of nonlinear terms, which is proved in Section 4 in a measure theoretic way having a background of Young measure theory.[2]

2 Backgrounds of the definition of a BV solution

In this section we review the definitions of a BV solution to (1.1) with (0.2), (0.3) that are discussed in [9, 10].

This equation is derived as the Euler-Lagrange equation of the action integral

$$(2.1)\qquad \int_0^T (\frac{1}{2}\int_\Omega |u_t(t,x)|^2 dx - \int_\Omega \sqrt{1+|\nabla u|^2}dx)dt.$$

The relaxiation A of the area functional is expressed as

$$A(u,\Omega) = \int_\Omega \sqrt{1+|\nabla u(x)|^2}dx + |D^s u|(\Omega)$$

[2]Note that varifold theory also has a background of Young measure theory.

(see [1, 7]). However this is not always Gâteaux differentiable on $BV(\Omega)$ and thus we cannot calculate $\frac{d}{d\varepsilon}A(u+\varepsilon\varphi,\Omega)|_{\varepsilon=0}$ directly. The area functional $A(u,\Omega)$ coincides with the n-dimensional Hausdorff measure of the reduced boundary $\partial^* E_u$ of the epigraph

$$E_u = \{(x,y); x \in \Omega, y > u(x)\}$$

(refer to [3], [7] for details about the reduced boundary), and we should only calculate a variation of $\mathcal{H}(\partial^* E_u)$. Noticing that the equation describes the longitudinal vibration, we could calculate the variation by the use of a one parameter family of diffeomorphisms of $U := \Omega \times \boldsymbol{R}$ each of which is written as $U \ni (x,y) \mapsto (x, y+\varepsilon\varphi(x,y)) \in U$, where ε is the parameter and φ is a given function on U. If $\varphi \in C_0^1(U)$, the function $\varepsilon \mapsto A(u+\varepsilon\varphi(x,u),\Omega)$ is differentiable and its derivative at $\varepsilon = 0$ is expressed by the use of $\nu_{E_u} := dD\chi_{E_u}/d|D\chi_{E_u}|$ (χ_{E_u} denotes the characteristic function of E_u and it belongs to $BV(U)$):

$$\frac{d}{d\varepsilon}A(u+\varepsilon\varphi(x,u))|_{\varepsilon=0} = \int_{\partial^* E_u}[-(\nabla_x\varphi\cdot\nu'_{E_u})\nu^{n+1}_{E_u} + |\nu'_{E_u}|^2\varphi_y]d\mathcal{H}^n \quad (\nu_{E_u} = (\nu'_{E_u}, \nu^{n+1}_{E_u}))$$

(compare to [9, Theorem 2.2]).

In [9], taking account of these facts, a BV solution to (1.1), (0.2), (0.3) is given as follows:

Definition 2.1 A function u is said to be a BV solution to (1.1), (0.2), (0.3) in $(0,T)\times\Omega$ if

i) $u \in L^\infty((0,T); BV(\Omega))$, $\quad u_t \in L^2((0,T)\times\Omega)$

ii) $\text{s-}\lim_{t\searrow 0} u(t) = u_0$ in $L^2(\Omega)$

iii) $\gamma u = 0$ for $\mathcal{L}^1$-a.e. $t \in (0,T)$

iv) for any $\varphi \in C_0^1([0,T)\times U)$,

$$\int_0^T\{-\int_\Omega u_t(\varphi_t(t,x,u)+\varphi_y(t,x,u)u_t)dx + \int_{\partial^* E_{u(t,\cdot)}}[-(\nabla_x\varphi\cdot\nu'_{E_{u(t,\cdot)}})\nu^{n+1}_{E_{u(t,\cdot)}}$$
$$+|\nu'_{E_{u(t,\cdot)}}|^2\varphi_y]d\mathcal{H}^n\}dt = \int_\Omega v_0(x)\varphi(0,x,u_0(x))dx.$$

Since the area functional A is convex, we can regard (1.1) as an evolution equation $u_{tt} + \partial A(u,\Omega) \ni 0$. It is proved in [9, Theorem A.1] that, if $\partial\Omega$ is of C^2 class, Definition 2.1 is equivalent to the definition of a weak solution to $u_{tt} + \partial A(u,\Omega) \ni 0$: putting

$$\mathcal{X} = \{\phi \in L^\infty((0,T); L^2(\Omega)\cap BV(\Omega)); \phi_t \in L^2((0,T)\times\Omega)\}$$

and

$$\mathcal{X}_0 = \{\phi \in \mathcal{X}; \gamma\phi = 0 \text{ for } \mathcal{L}^1 \text{ -a.e. } t \in (0,T)\},$$

we define

Definition 2.2 A function u is said to be a BV solution to (1.1), (0.2), (0.3) in $(0,T)\times\Omega$ if i), ii), iii), and

iv)' for any $\phi\in C_0^0([0,T);L^2(\Omega))\cap\mathcal{X}_0$,

$$\int_0^T\{A(u+\phi,\Omega)-A(u,\Omega)\}dt\geq\int_0^T\int_\Omega u_t\phi_t(t,x)dxdt+\int_\Omega v_0(x)\phi(0,x)dx.$$

But in [10] it is pointed out that the appropriate weak formulation of Dirichlet condition (0.3) is not to suppose the trace vanishes but to replace $A(u,\Omega)$ with

$$A(u,\overline{\Omega})=A(u,\Omega)+\int_{\partial\Omega}|\gamma u(x)|d\mathcal{H}^{n-1}.$$

Thus in [10] a solution is defined as

Definition 2.3 A function u is said to be a BV solution to (1.1), (0.2), (0.3) in $(0,T)\times\Omega$ if and only if i), ii), and

v) for any $\phi\in C_0^0([0,T);L^2(\Omega))\cap\mathcal{X}$,

$$\int_0^T\{A(u+\phi,\overline{\Omega})-A(u,\overline{\Omega})\}dt\geq\int_0^T\int_\Omega u_t\phi_t(t,x)dxdt+\int_\Omega v_0(x)\phi(0,x)dx.$$

Further in [10] another definition is presented and proved that it is equivalent to Definition 2.3 if $\partial\Omega$ is of C^2 class (compare to Definitions 2.1 and 2.2).

Definition 2.4 A function u is said to be a BV solution to (1.1), (0.2), (0.3) in $(0,T)\times\Omega$ if and only if i), ii),

v)$_1$' for any $\varphi\in C_0^1([0,T)\times U)$,

$$\begin{aligned}\int_0^T\{-\int_\Omega u_t(\varphi_t(t,x,u)+\varphi_y(t,x,u)u_t)dx+\int_{\partial^*E_{u(t,\cdot)}}[-(\nabla_x\varphi\cdot\nu'_{E_{u(t,\cdot)}})\nu^{n+1}_{E_{u(t,\cdot)}}\\+|\nu'_{E_{u(t,\cdot)}}|^2\varphi_y]d\mathcal{H}^n\}dt=\int_\Omega v_0(x)\varphi(0,x,u_0(x))dx\end{aligned}$$

v)$_2$' for any $\psi\in C_0^1([0,T))$,

$$\begin{aligned}\int_0^T\{-\int_\Omega u_t(\psi'(t)u+\psi(t)u_t)dx+\psi(t)\int_{\partial^*E_{u(t,\cdot)}}|\nu'_{E_{u(t,\cdot)}}|^2d\mathcal{H}^n\\+\psi(t)\int_{\partial\Omega}|\gamma u|d\mathcal{H}^{n-1}\}dt=\psi(0)\int_\Omega v_0(x)u_0(x)dx.\end{aligned}$$

(Note that v)$_1$' of Definition 2.4 is the same condition as iv) of Definition 2.1.)

Looking at the proof of the equivalence between Definitions 2.3 and 2.4 carefully, we find that it is obtained by testing only smooth functions and u itself. Thus, in fact, if $\partial\Omega$ is of C^2 class, Definitions 2.3 and 2.4 are also equivalent to

Definition 2.5 A function u is said to be a BV solution to (1.1), (0.2), (0.3) in $(0,T)\times\Omega$ if and only if i), ii),

v)$_1$" for any $\phi\in C_0^1([0,T)\times\Omega)$,

$$\int_0^T\{-\int_\Omega u_t\phi_t(t,x)dx+\int_\Omega\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\nabla\phi(t,x)dx\}dt=\int_\Omega v_0(x)\phi(0,x)dx$$

v)$_2$" for any $\psi\in C_0^1([0,T))$,

$$\int_0^T\{-\int_\Omega u_t(\psi'(t)u+\psi(t)u_t)dx+\psi(t)\int_\Omega\frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}}dx+\psi(t)|D^su|(\Omega)$$
$$+\psi(t)\int_{\partial\Omega}|\gamma u|d\mathcal{H}^{n-1}\}dt=\psi(0)\int_\Omega v_0(x)u_0(x)dx.$$

Implication relations among these definitions are as follows:

$$\text{Definition 2.2}\ \begin{array}{c}\Longrightarrow\ \text{Definition 2.1}\ \Longrightarrow\\ \Longrightarrow\ \text{Definition 2.3}\ \Longrightarrow\end{array}\ \text{Definition 2.4}\ \Longrightarrow\ \text{Definition 2.5.}$$

If $\partial\Omega$ is of C^2 class, the converses except for 2.3 $\Rightarrow$ 2.2 and 2.4 $\Rightarrow$ 2.1 also hold.

Clearly Definition 1.1 is a vectorial generalization of Definition 2.5. Definitions 2.2 and 2.3 are based on the convexity of A, and we are unable to employ them for our problem since our functional is not in general convex. Thus the most appropriate definition is a generalization of Definition 2.4. However it would be hard to treat for vectorial cases and hence we employ Definition 2.5 for the generalization.

3 Apploximate solutions and our main theorem

Suppose that $u_0=(u_0^1,u_0^2,\dots,u_0^N)\in[L^2(\Omega)\cap BV(\Omega)]^N$ and $v_0=(v_0^1,v_0^2,\dots,v_0^N)\in[L^2(\Omega)]^N$. For a positive number h we construct a sequence $\{u_\ell^j;\ell=-1,0,1,\dots,\ j=1,2,\dots,N\}$ in the following way. For $\ell=0$ we let u_0^j be as above and for $\ell=-1$ we set $u_{-1}^j=u_0^j-hv_0^j$. Suppose that $u_{\ell-1}^j$ ($\ell\geq1$, $j=1,2,\dots,N$) are already defined. Then we define u_ℓ^1 as the minimizer of the functional

$$\mathcal{F}_\ell^1(v)=\frac{1}{2}\int_\Omega\frac{|v-2u_{\ell-1}^1+u_{\ell-2}^1|^2}{h^2}dx+J(v,u_{\ell-1}^2,\dots,u_{\ell-1}^N,\overline{\Omega})$$

in $L^2(\Omega)\cap BV(\Omega)$. Suppose that u_ℓ^{j-1} ($j=2,\dots,N$) are defined. Then we define u_ℓ^j as the minimizer of the functional

$$\mathcal{F}_\ell^j(v)=\frac{1}{2}\int_\Omega\frac{|v-2u_{\ell-1}^j+u_{\ell-2}^j|^2}{h^2}dx+J(u_\ell^1,\dots,u_\ell^{j-1},v,u_{\ell-1}^{j+1},\dots,u_{\ell-1}^N,\overline{\Omega})$$

in $L^2(\Omega)\cap BV(\Omega)$. Now we put

$$u_\ell={}^t(u_\ell^1,u_\ell^2,\dots,u_\ell^N)\in[L^2(\Omega)\cap BV(\Omega)]^N.$$

First we show the energy inequality

$$\frac{1}{2}\int_{\Omega}\frac{|u_\ell-u_{\ell-1}|^2}{h^2}dx+J(u_\ell,\overline{\Omega})\le\frac{1}{2}\int_{\Omega}|v_0|^2dx+J(u_0,\overline{\Omega}). \tag{3.1}$$

Moreover, putting

$$u_\ell^{(j)}={}^t(u_\ell^1,\ldots,u_\ell^{j-1},u_\ell^j,u_{\ell-1}^{j+1},\ldots,u_{\ell-1}^N),$$

we have the following proposition.

Proposition 3.1 *For each $j=1,2,\ldots,N$ and $\ell=1,2,\ldots$*

$$\frac{1}{2}\int_{\Omega}\frac{|u_\ell^{(j)}-u_{\ell-1}^{(j)}|^2}{h^2}dx+J(u_\ell^{(j)},\overline{\Omega})\le\frac{1}{2}\int_{\Omega}|v_0|^2dx+J(u_0,\overline{\Omega}).$$

Proof. For the sake of simplicity we write

$$J_\ell^j(v,\overline{\Omega})=J(u_\ell^1,\ldots,u_\ell^{j-1},v,u_{\ell-1}^{j+1},\ldots,u_{\ell-1}^N,\overline{\Omega}).$$

By the minimality of $\mathcal{F}_\ell^j(u_\ell^j)$ we have

$$\begin{aligned}\mathcal{F}_\ell^j(u_\ell^j)&=\frac{1}{2}\int_{\Omega}\frac{|u_\ell^j-2u_{\ell-1}^j+u_{\ell-2}^j|^2}{h^2}dx+J(u_\ell,\overline{\Omega})\le\mathcal{F}_\ell^j((1-\theta)u_\ell^j+\theta u_{\ell-1}^j)\\&=\frac{1}{2}\int_{\Omega}\frac{|(1-\theta)(u_\ell^j-u_{\ell-1}^j)-u_{\ell-1}^j+u_{\ell-2}^j|^2}{h^2}dx+J_\ell^j((1-\theta)u_\ell^j+\theta u_{\ell-1}^j,\overline{\Omega})\end{aligned} \tag{3.2}$$

for $0\le\theta\le1$. By an easy calculus we obtain

$$\begin{aligned}&|u_\ell^j-2u_{\ell-1}^j+u_{\ell-2}^j|^2-|(1-\theta)(u_\ell^j-u_{\ell-1}^j)-u_{\ell-1}^j+u_{\ell-2}^j|^2\\&\qquad\le\theta((1-\theta)|u_\ell^j-u_{\ell-1}^j|^2-|u_{\ell-1}^j-u_{\ell-2}^j|^2).\end{aligned}$$

This and (3.2) imply

$$\begin{aligned}&\theta\frac{1}{2}\int_{\Omega}\frac{(1-\theta)|u_\ell^j-u_{\ell-1}^j|^2}{h^2}dx+J_\ell^j(u_\ell^j,\overline{\Omega})\\&\qquad\le\theta\frac{1}{2}\int_{\Omega}\frac{|u_{\ell-1}^j-u_{\ell-2}^j|^2}{h^2}dx+J_\ell^j((1-\theta)u_\ell^j+\theta u_{\ell-1}^j,\overline{\Omega}).\end{aligned} \tag{3.3}$$

Since f is quasiconvex and thus rank-one convex, J_ℓ^j is convex. Hence the second term of the right hand side of (3.3) is less than $(1-\theta)J_\ell^j(u_\ell^j,\overline{\Omega})+\theta J_\ell^j(u_{\ell-1}^j,\overline{\Omega})$ and then we have

$$\theta\frac{1}{2}\int_{\Omega}\frac{(1-\theta)|u_\ell^j-u_{\ell-1}^j|^2}{h^2}dx+\theta J_\ell^j(u_\ell^j,\overline{\Omega})\le\theta\frac{1}{2}\int_{\Omega}\frac{|u_{\ell-1}^j-u_{\ell-2}^j|^2}{h^2}dx+\theta J_\ell^j(u_{\ell-1}^j,\overline{\Omega}).$$

Multiplying θ^{-1} to the both side and letting $\theta\searrow0$, we have

$$\frac{1}{2}\int_{\Omega}\frac{|u_\ell^j-u_{\ell-1}^j|^2}{h^2}dx+J_\ell^j(u_\ell^j,\overline{\Omega})\le\frac{1}{2}\int_{\Omega}\frac{|u_{\ell-1}^j-u_{\ell-2}^j|^2}{h^2}dx+J_\ell^j(u_{\ell-1}^j,\overline{\Omega}). \tag{3.4}$$

Noting that

$$J_\ell^N(u_\ell^N,\overline{\Omega}) = J(u_\ell,\overline{\Omega}),\ J_\ell^1(u_{\ell-1}^1,\overline{\Omega}) = J(u_{\ell-1},\overline{\Omega}),\ \text{and } J_\ell^j(u_{\ell-1}^j,\overline{\Omega}) = J_\ell^{j-1}(u_\ell^{j-1},\overline{\Omega}),$$

we have by (3.4)

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \frac{|u_\ell - u_{\ell-1}|^2}{h^2}dx + J(u_\ell,\overline{\Omega}) = \frac{1}{2}\sum_{j=1}^N \int_\Omega \frac{|u_\ell^j - u_{\ell-1}^j|^2}{h^2}dx + J_\ell^N(u_\ell^N,\overline{\Omega})\\
\le\ & \frac{1}{2}\sum_{j=1}^{N-1}\int_\Omega \frac{|u_\ell^j - u_{\ell-1}^j|^2}{h^2}dx + \frac{1}{2}\int_\Omega \frac{|u_{\ell-1}^N - u_{\ell-2}^N|^2}{h^2}dx + J_\ell^N(u_{\ell-1}^N,\overline{\Omega})\\
=\ & \frac{1}{2}\sum_{j=1}^{N-1}\int_\Omega \frac{|u_\ell^j - u_{\ell-1}^j|^2}{h^2}dx + \frac{1}{2}\int_\Omega \frac{|u_{\ell-1}^N - u_{\ell-2}^N|^2}{h^2}dx + J_\ell^{N-1}(u_\ell^{N-1},\overline{\Omega})\\
\le\ & \frac{1}{2}\sum_{j=1}^{N-2}\int_\Omega \frac{|u_\ell^j - u_{\ell-1}^j|^2}{h^2}dx + \sum_{j=N-1}^{N}\frac{1}{2}\int_\Omega \frac{|u_{\ell-1}^j - u_{\ell-2}^j|^2}{h^2}dx + J_\ell^{N-1}(u_{\ell-1}^{N-1},\overline{\Omega})\\
\le\ & \cdots\\
\le\ & \frac{1}{2}\int_\Omega \frac{|u_\ell^1 - u_{\ell-1}^1|^2}{h^2}dx + \sum_{j=2}^{N}\frac{1}{2}\int_\Omega \frac{|u_{\ell-1}^j - u_{\ell-2}^j|^2}{h^2}dx + J_\ell^2(u_{\ell-1}^2,\overline{\Omega})\\
=\ & \frac{1}{2}\int_\Omega \frac{|u_\ell^1 - u_{\ell-1}^1|^2}{h^2}dx + \sum_{j=2}^{N}\frac{1}{2}\int_\Omega \frac{|u_{\ell-1}^j - u_{\ell-2}^j|^2}{h^2}dx + J_\ell^1(u_\ell^1,\overline{\Omega})\\
\le\ & \sum_{j=1}^{N}\frac{1}{2}\int_\Omega \frac{|u_{\ell-1}^j - u_{\ell-2}^j|^2}{h^2}dx + J_\ell^1(u_{\ell-1}^1,\overline{\Omega}) = \frac{1}{2}\int_\Omega \frac{|u_{\ell-1} - u_{\ell-2}|^2}{h^2}dx + J(u_{\ell-1},\overline{\Omega}).
\end{aligned}$$

Since $J_\ell^j(u_\ell^j,\overline{\Omega}) = J(u_\ell^{(j)},\overline{\Omega})$, we have the conclusion by induction on ℓ. Q.E.D.

Remark. Clearly (3.1) is the case of $j = N$ of Proposition 3.1.

Next we define approximate solutions

$$u^h(t,x) = {}^t(u^{h,1},u^{h,2},\dots u^{h,N}) \quad \text{and} \quad \overline{u}^h(t,x) = {}^t(\overline{u}^{h,1},\overline{u}^{h,2},\dots \overline{u}^{h,N})$$

for $(t,x) \in (0,\infty)\times\Omega$ as follows: for $(\ell-1)h < t \le \ell h$

$$u^h(t,x) = \frac{t-(\ell-1)h}{h}u_\ell(x) + \frac{\ell h - t}{h}u_{\ell-1}(x) \tag{3.5}$$

and

$$\overline{u}^h(t,x) = u_\ell(x). \tag{3.6}$$

Then (3.1) shows, for each $t \in \bigcup_{\ell=0}^\infty ((\ell-1)h,\ell h)$,

$$\frac{1}{2}\int_\Omega |u_t^h(t,x)|^2dx + J(\overline{u}^h(t,\cdot),\overline{\Omega}) \le \frac{1}{2}\int_\Omega |v_0|^2dx + J(u_0,\overline{\Omega})$$

Replacing u_ℓ and $u_{\ell-1}$ in (3.5) and (3.6) with $u_\ell^{(j)}$ and $u_{\ell-1}^{(j)}$, respectively, we define $u^{h,(j)}$ and $\overline{u}^{h,(j)}$, and we more have by Lemma 3.1, for each $t \in \bigcup_{\ell=0}^\infty ((\ell-1)h,\ell h)$,

$$\frac{1}{2}\int_\Omega |u_t^{h,(j)}(t,x)|^2dx + J(\overline{u}^{h,(j)}(t,\cdot),\overline{\Omega}) \le \frac{1}{2}\int_\Omega |v_0|^2dx + J(u_0,\overline{\Omega}). \tag{3.7}$$

Note that

$$\overline{u}^{h,(j)}(t,x) = {}^t(\overline{u}^{h,1}(t), \overline{u}^{h,2}(t), \ldots \overline{u}^{h,j}(t), \overline{u}^{h,j+1}(t-h), \ldots, \overline{u}^{h,N}(t-h)).$$

By the use of (3.7) we can obtain the following theorem (compare to the proof of [9, Theorem 3.3]).

Proposition 3.2 *Let T be any positive number. It holds that, for each $j = 1, 2, \ldots, N$,*
1) $\{\| u_t^{h,(j)} \|_{L^\infty((0,\infty);L^2(\Omega))}\}$ *is uniformly bounded with respect to h*
2) $\{\| u^{h,(j)} \|_{L^\infty((0,T);L^2(\Omega)\cap BV(\Omega))}\}$ *is uniformly bounded with respect to h*
3) $\{\| \overline{u}^{h,(j)} \|_{L^\infty((0,T);L^2(\Omega)\cap BV(\Omega))}\}$ *is uniformly bounded with respect to h.*
Then there exist a sequence $\{h_m\}$ with $h_m \to 0$ as $m \to \infty$ and a function u such that
4) $\overline{u}^{h_m,(j)}$ *converges to u as $m \to \infty$ weakly star in $[L^\infty((0,T); L^2(\Omega))]^N$*
5) $u_t^{h_m,(j)}$ *converges to u_t as $m \to \infty$ weakly star in $[L^\infty((0,\infty); L^2(\Omega))]^N$*
6) $u^{h_m,(j)}$ *converges to u as $m \to \infty$ strongly in $[L^p((0,T) \times \Omega)]^N$ for each $1 \le p < 1^*$*
7) $\overline{u}^{h_m,(j)}$ *converges to u as $m \to \infty$ strongly in $[L^p((0,T) \times \Omega)]^N$ for each $1 \le p < 1^*$*
8) $u \in [L^\infty((0,\infty); BV(\Omega))]^N$
9) *for $\mathcal{L}^1$-a.e. $t \in (0,\infty)$, $D\overline{u}^{h_m,(j)}(t,\cdot)$ converges to $Du(t,\cdot)$ as $m \to \infty$ in the sense of distributions*
10) $\operatorname{s-lim}_{t\searrow 0} u(t) = u_0$ *in* $[L^2(\Omega)]^N$.

Remark. In the sequel $\{u^{h_m}\}$ and $\{\overline{u}^{h_m}\}$ are often denoted by $\{u^h\}$ and $\{\overline{u}^h\}$ for simplicity.

Our main theorem is as follows (assumption (A6) is stated in Section 4):

Theorem 3.3 *Suppose that f satisfies* (A1) $\sim$ (A6). *Let T be a positive number. If u as in* Proposition 3.2 *satisfies the energy conservation law*

$$(3.8)\qquad \frac{1}{2}\int_\Omega |u_t(t,x)|^2 dx + J(u(t,\cdot),\overline{\Omega}) = \frac{1}{2}\int_\Omega |v_0(x)|^2 dx + J(u_0,\overline{\Omega})$$

for $\mathcal{L}^1$-a.e. $t \in (0,T)$, then u is a BV solution to (0.1)–(0.3) *in* $(0,T) \times \Omega$.

Let $\iota_{j,\varepsilon}$ denote the N by N matrix defined by

$$\iota_{j,\varepsilon} = \operatorname{diag}(1, \ldots 1, \underbrace{1+\varepsilon}_{\text{j th}}, 1, \ldots, 1).$$

Using assumptions (A1) $\sim$ (A5), we can show the following lemma (in fact assumption (A2) is not necessary for this lemma). The proof of this lemma is not so difficult and thus we omit it.

Lemma 3.4 1) *The limitsup of* (1.4) *is in fact a limit. Furthermore the limit is uniform with respect to p in a compact subset of $\boldsymbol{R}^{nN}$*

2) $\lim_{\rho\searrow 0} f_p(\frac{p}{\rho}) : p = f_\infty(p)$

3) $\lim_{\varepsilon\to 0} \sum_{j=1}^N \frac{f_\infty(\iota_{j,\varepsilon} p) - f_\infty(p)}{\varepsilon} = f_\infty(p).$

Proof of Theorem 3.3. Proposition 3.2 5) and 8) imply i) and 10) implies ii). Thus in order to obtain the conclusion we should show iii) and iv) of Definition 1.1.

By Proposition 3.2 5) we have, for each $j = 1, 2, \dots, N$,

$$\liminf_{h\searrow 0} \int_0^T \int_\Omega |u_t^{h,(j)}(t,x)|^2 dxdt \ge \int_0^T \int_\Omega |u_t(t,x)|^2 dxdt.$$

Since J is lower semicontinuous by (A2), we more have by Proposition 3.2 7) and 8), for $\mathcal{L}^1$-a.e. $t \in (0,T)$,

$$\liminf_{h\searrow 0} J(\overline{u}^{h,(j)}(t,\cdot), \overline{\Omega}) \ge J(u(t,\cdot), \overline{\Omega}). \tag{3.9}$$

Thus, integrating energy inequality (3.7) and energy conservation law (3.8) over $(0,T)$, we have

$$\lim_{h\searrow 0} \int_0^T \int_\Omega |u_t^{h,(j)}(t,x)|^2 dxdt = \int_0^T \int_\Omega |u_t(t,x)|^2 dxdt \tag{3.10}$$

(and $\lim_{h\searrow 0} \int_0^T J(\overline{u}^{h,(j)}, \overline{\Omega})dt = \int_0^T J(u, \overline{\Omega})dt$). In particular $\{u_t^{h,(j)}\}$ converges to u_t strongly in $L^2((0,T)\times\Omega)$, and hence

$$\lim_{h\searrow 0} \int_\Omega |u_t^{h,(j)}(t,x)|^2 dx = \int_\Omega |u_t(t,x)|^2 dx \tag{3.11}$$

for $\mathcal{L}^1$-a.e. $t \in (0,T)$. By (3.7), (3.8), and (3.9) we also obtain, for $\mathcal{L}^1$-a.e. $t \in (0,T)$,

$$\lim_{h\searrow 0} J(\overline{u}^{h,(j)}(t,\cdot), \overline{\Omega}) = J(u(t,\cdot), \overline{\Omega}). \tag{3.12}$$

Since u_ℓ^j is the minimizer of $\mathcal{F}_\ell^j$, we have

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon} \mathcal{F}_\ell^j (u_\ell^j + \varepsilon\varphi)|_{\varepsilon=0} \\ &= \int_\Omega \frac{u_\ell^j(x) - 2u_{\ell-1}^j(x) + u_{\ell-2}^j(x)}{h^2} \varphi(x)dx + \frac{d}{d\varepsilon} J_\ell^j(u_\ell^j + \varepsilon\varphi, \overline{\Omega})|_{\varepsilon=0} \end{aligned}$$

for any $\varphi \in C_0^1(\Omega)$. Putting

$$\tilde{\varphi} = (0, \dots 0, \underset{\text{j th}}{\varphi}, 0, \dots, 0),$$

we have by Federer-Vol'pert's theorem (Theorem 3.78 of [1]) $S_{u_\ell^{(j)}+\varepsilon\tilde{\varphi}} = S_{u_\ell^{(j)}}$ and $D^s(u_\ell^{(j)} + \varepsilon\tilde{\varphi}) = D^s u_\ell^{(j)}$. Hence by (1.3) and (1.5)

$$\frac{d}{d\varepsilon} J_\ell^j (u_\ell^j + \varepsilon\varphi, \overline{\Omega})|_{\varepsilon=0} = \int_\Omega f_{p^j}(\nabla u_\ell^{(j)}) \nabla\varphi(x) dx.$$

Noting that, for $(\ell-1)h < t < \ell h$, $(\partial u^h/\partial t)(t) = (u_\ell - u_{\ell-1})/h$, we have, for any $\varphi = (\varphi^1, \varphi^2, \dots, \varphi^N) \in [C_0^1(\Omega)]^N$ and any $\psi \in C_0^1([0,T))$,

$$\int_0^T \psi(t) [\int_\Omega \frac{u_t^h(t,x) - u_t^h(t-h,x)}{h} \varphi(x)dx + \sum_{j=1}^N \int_\Omega f_{p^j}(\nabla\overline{u}^{h,(j)}(t)) \nabla\varphi^j(x) dx] dt = 0. \tag{3.13}$$

Thus, if we show, as $h \to 0$, passing to a subsequence if necessary,

$$\begin{aligned}(3.14)\qquad &\int_0^T \psi(t)\int_\Omega \frac{u_t^h(t,x)-u_t^h(t-h,x)}{h}\varphi(x)dxdt\\ &\longrightarrow -\int_0^T \psi_t(t)\int_\Omega u_t(t,x)\varphi(x)dxdt - \psi(0)\int_\Omega v_0(x)\varphi(x)dx\end{aligned}$$

and for each $j = 1, 2, \ldots, N$

$$(3.15)\qquad \int_0^T \psi(t)\int_\Omega f_{p^j}(\nabla\overline{u}^{h,(j)})\nabla\varphi^j(x)dxdt \longrightarrow \int_0^T \psi(t)\int_\Omega f_{p^j}(\nabla u)\nabla\varphi^j(x)dxdt,$$

then we have iii) of Definition 1.1 by (3.13). Proofs of (3.14) and (3.15) are presented later.

By the minimality of $\mathcal{F}_\ell^j(u_\ell^j)$ again we have

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}\mathcal{F}_\ell^j(u_\ell^j + \varepsilon u_\ell^j)|_{\varepsilon=0}\\ &= \int_\Omega \frac{u_\ell^j - 2u_{\ell-1}^j + u_{\ell-2}^j}{h^2}u_\ell^j dx + \frac{d}{d\varepsilon}J_\ell^j(u_\ell^j + \varepsilon u_\ell^j, \overline{\Omega})|_{\varepsilon=0}.\end{aligned}$$

Since the functional J_ℓ^j is convex, we have for each $\varepsilon > 0$ (resp. $\varepsilon < 0$)

$$\varepsilon^{-1}(J_\ell^j(u_\ell^j + \varepsilon u_\ell^j, \overline{\Omega}) - J_\ell^j(u_\ell^j, \overline{\Omega})) \geq \frac{d}{d\varepsilon}J_\ell^j(u_\ell^j + \varepsilon u_\ell^j, \overline{\Omega})|_{\varepsilon=0} \quad \text{(resp. } \leq).$$

Thus we find

$$0 \leq J_\ell^j(u_\ell^j + \varepsilon u_\ell^j, \overline{\Omega}) - J_\ell^j(u_\ell^j, \overline{\Omega}) + \varepsilon\int_\Omega \frac{u_\ell^j - 2u_{\ell-1}^j + u_{\ell-2}^j}{h^2}u_\ell^j dx,$$

which immediately implies for any $T > 0$, for any $\psi \in C_0^1([0,T))$, and for any $\varepsilon \neq 0$

$$\begin{aligned}(3.16)\qquad &\varepsilon\int_0^T \psi(t)\{\int_\Omega \frac{u_t^h(t,x)-u_t^h(t-h,x)}{h}\overline{u}^h(t,x)dx\\ &\qquad + \sum_{j=1}^N [J(\iota_{j,\varepsilon}\overline{u}^{h,(j)}, \overline{\Omega}) - J(\overline{u}^{h,(j)}, \overline{\Omega})]\}dt \geq 0.\end{aligned}$$

Suppose that we have, as $h \to 0$, passing to a subsequence if necessary,

$$\begin{aligned}(3.17)\qquad &\int_0^T \psi(t)\int_\Omega \frac{u_t^h(t,x)-u_t^h(t-h,x)}{h}\overline{u}^h(t,x)dxdt\\ &\longrightarrow \int_0^T\{-\int_\Omega u_t(\psi'(t)u + \psi(t)u_t)dx\}dt - \psi(0)\int_\Omega v_0(x)u_0(x)dx\end{aligned}$$

and

$$(3.18)\qquad \int_0^T \psi(t)J(\iota_{j,\varepsilon}\overline{u}^{h,(j)}, \overline{\Omega})dt \longrightarrow \int_0^T \psi(t)J(\iota_{j,\varepsilon}u, \overline{\Omega})dt.$$

Then (3.16) implies

$$\begin{aligned}(3.19)\qquad &\int_0^T\{-\int_\Omega u_t(\psi'(t)u + \psi(t)u_t)dx\}dt - \psi(0)\int_\Omega v_0(x)u_0(x)dx\\ &\qquad + \sum_{j=1}^N [J(\iota_{j,\varepsilon}u, \overline{\Omega}) - J(u, \overline{\Omega})]\}dt \geq 0.\end{aligned}$$

It follows from (1.3), (1.5), and Lemma 3.4 3) that

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\sum_{j=1}^{N}[J(\iota_{j,\varepsilon}u,\overline{\Omega})-J(u,\overline{\Omega})]=\int_\Omega f_p(\nabla u):\nabla u dx$$
$$+\int_\Omega f_\infty(\frac{dD^su}{d|D^su|})d|D^su|+\int_{\partial\Omega}f_\infty(\gamma u\otimes\vec{n})d\mathcal{H}^{n-1}.$$

Hence, multiplying ε^{-1} to the both side of (3.19) and letting $\varepsilon\searrow 0$ and $\varepsilon\nearrow 0$, we obtain iv) of Definition 1.1.

Now it remains to prove (3.14), (3.15), (3.17), (3.18). In this section, accepting (3.15) and (3.18), we conclude the proof of Theorem 3.3 by showing (3.14) and (3.17). Proofs of (3.15) and (3.18) are left to the next section.

Let ϕ be either $\psi\varphi$ or $\psi\overline{u}^h$. First we rewrite

$$\begin{aligned}
&\int_0^T\int_\Omega\frac{u_t^h(t,x)-u_t^h(t-h,x)}{h}\phi(t,x)dxdt \qquad (3.20)\\
=\ &\int_0^\infty\int_\Omega\frac{u_t^h(t,x)-u_t^h(t-h,x)}{h}\phi(t,x)dxdt\\
=\ &\int_0^\infty\int_\Omega\frac{u_t^h(t,x)}{h}\phi(t,x)dxdt-\int_{-h}^\infty\int_\Omega\frac{u_t^h(s,x)}{h}\phi(s+h,x)dxds\\
=\ &-\{\int_0^\infty\int_\Omega u_t^h(t,x)\frac{\phi(t+h,x)-\phi(t,x)}{h}dxdt\\
&\quad+\frac{1}{h}\int_{-h}^0\int_\Omega u_t^h(s,x)\phi(s+h,x)dxds\}\\
=:\ &-(I+II).
\end{aligned}$$

Noting that $u_t^h(s,x)=v_0(x)$ for $-h<s\le 0$, we have

$$II=\int_\Omega v_0(x)\frac{1}{h}\int_{-h}^0\phi(s+h,x)dsdx=\int_\Omega v_0(x)\frac{1}{h}\int_0^h\phi(t,x)dtdx. \qquad (3.21)$$

In case $\phi=\psi\varphi$ ($\psi\in C_0^1([0,T))$, $\varphi\in C_0^1(\Omega)$), since

$$\frac{\psi(t+h)-\psi(t)}{h}\to\psi_t(t)$$

strongly in $L^\infty(0,T)$ and

$$\int_{-h}^0\psi(s+h)ds\to\psi(0),$$

we have (3.14) by Proposition 3.2 5). In case $\phi=\psi\overline{u}^h$ ($\psi\in C_0^1([0,T))$), we first have by (3.21), noting further that $\overline{u}^h(t,x)=u_1(x)$ for $0<t\le h$,

$$II=\frac{1}{h}\int_0^h\psi(t)dt\int_\Omega v_0(x)u_1(x)dx.$$

Since, for $0<t<h$,

$$u_1(x)=u_0(x)+h\frac{u_1(x)-u_0(x)}{h}=u_0(x)+hu_t^h(t,x),$$

we have by Proposition 3.2 1)

$$\lim_{h\searrow 0} II = \psi(0)\int_\Omega v_0(x)u_0(x)dx. \tag{3.22}$$

On the other hand we have

$$\begin{aligned}
I &= \int_0^\infty \int_\Omega \frac{\psi(t+h)\overline{u}^h(t+h,x)-\psi(t)\overline{u}^h(t,x)}{h}dxdt \\
&= \int_0^\infty \frac{\psi(t+h)-\psi(t)}{h}\int_\Omega \overline{u}^h(t+h,x)dxdt + \int_0^\infty \psi(t)\int_\Omega \frac{\overline{u}^h(t+h,x)-\overline{u}^h(t,x)}{h}dxdt \\
&= \int_0^\infty \int_0^1 \psi_t(t+\theta h)d\theta \int_\Omega \overline{u}^h(t+h,x)dxdt + \int_0^\infty \psi(t)\int_\Omega u_t^h(t+h,x)dxdt.
\end{aligned}$$

We see that

$$\int_0^1 \psi_t(t+\theta h)d\theta \to \psi_t(t).$$

By (3.10), $\{u_t^h\}$ converges to u_t strongly in $L^2((0,T)\times\Omega)$. Let T' be any number with $0<T'<T$. If $0<h<T-T'$, we have

$$\| u_t^h(\cdot+h)-u_t(\cdot+h)\|_{L^2((0,T')\times\Omega)} = \| u_t^h - u_t\|_{L^2((h,T'+h)\times\Omega)} \le \| u_t^h - u_t\|_{L^2((0,T)\times\Omega)},$$

the right hand side of which converges to 0 as $h\to 0$. It follows from Lusin's theorem that, as $h\to 0$,

$$\| u_t(\cdot+h)-u_t\|_{L^2((0,T')\times\Omega)} \to 0.$$

Thus, writing

$$\begin{aligned}
&\| u_t^h(\cdot+h)-u_t\|_{L^2((0,T')\times\Omega)} \\
&\quad \le \| u_t^h(\cdot+h)-u_t(\cdot+h)\|_{L^2((0,T')\times\Omega)} + \| u_t(\cdot+h)-u_t\|_{L^2((0,T')\times\Omega)},
\end{aligned}$$

we obtain that $u_t^h(\cdot+h)\to u_t$ strongly in $L^2((0,T')\times\Omega)$. Noting that the support of φ with respect to the t variable is a compact subset of $[0,T)$, we have

$$\lim_{h\searrow 0} I = \int_0^\infty \int_\Omega u_t(\psi_t(t)u+\psi(t)u_t)dxdt. \tag{3.23}$$

Now (3.17) follows from (3.20), (3.22), and (3.23).

Thus the proof is complete except for proofs of (3.15) and (3.18). Q.E.D.

4 Radon measures in $\overline{\Omega}\times\overline{S}_+$

Let μ be a $\boldsymbol{R}^m$ valued Radon measure. Then we write its total variation as $|\mu|$ and the Radon-Nikodym derivative of μ with respect to $|\mu|$ as $\vec{\mu}$. In particular, $\mu = |\mu| \llcorner \vec{\mu}$.

For $v\in[BV(\Omega)]^N$ we define an $\boldsymbol{R}^{nN+1}$ valued Radon measure μ_v by

$$\mu_v = {}^t(-Dv,\mathcal{L}^n).$$

For an open set $A\subset\Omega$, total variation $|\mu_v|$ is given by

$$|\mu_v|(A) = \sup\{\int_\Omega (g_0+v\,\mathrm{div}g)dx; (g_0,g)\in C^1(\Omega,\boldsymbol{R}^{nN+1}),\ |g_0|^2+|g|^2\le 1\}$$

In this article, for the sake of simplicity, we write $S_+^{nN+1} = S_+$:

$$S_+ = \{\vec{s} = (s^1, \cdots, s^{nN+1}) \in S^{nN};\, s^{nN+1} > 0\}.$$

We also write

$$S_0 = \{\vec{s} = (s^1, \cdots, s^{nN+1}) \in S^n;\, s^{nN+1} = 0\}.$$

Then $\overline{S}_+ = S_+ \cup S_0$. Given a Radon measure λ in $\overline{\Omega} \times \overline{S}_+$, we let $|\lambda|$ denote a Radon measure on $\overline{\Omega}$ defined by

$$|\lambda|(A) = \lambda(A \times \overline{S}_+) \qquad \text{for a Borel set } A \subset \overline{\Omega}.$$

Clearly this notation is an analogy with that of a total variations of a vector valued Radon measure. In particular, letting λ be a Radon measure in $\overline{\Omega} \times \overline{S}_+$ defined as, for a BV function $v \in [BV(\Omega)]^N$,

$$\int_{\overline{\Omega}\times\overline{S}_+} \beta(x, \vec{s})d\lambda = \int_{\overline{\Omega}} \beta(x, \vec{\mu_v}(x))d|\mu_v| \qquad (\beta \in C^0(\overline{\Omega} \times \overline{S}_+)), \tag{4.1}$$

then we have $|\lambda| = |\mu_v|$. For each Radon measure λ in $\overline{\Omega} \times \overline{S}_+$, there exists a probability Radon measure $\nu_{\lambda,x}$ on $\overline{S}_+$ for $|\lambda|$-a.e. $x \in \overline{\Omega}$ such that

$$\int_{\overline{\Omega}\times\overline{S}_+} \beta(x, \vec{s})d\lambda = \int_{\overline{\Omega}} (\int_{\overline{S}_+} \beta(x, \vec{s})d\nu_{\lambda,x})d|\lambda| \qquad (\beta \in C^0(\overline{\Omega} \times \overline{S}_+))$$

(for example, Theorem 10 of page 14 of [2]). Using these notations, we often write $\lambda = |\lambda| \otimes \nu_{\lambda,x}$. In particular, if λ is as in (4.1), then $\lambda = |\mu_v| \otimes \delta_{\vec{\mu}_v(x)}$.

We define a function F on $\overline{S}_+$ as follows: for $\vec{s} = (s', s^{nN+1}) \in \overline{S}_+$,

$$F(\vec{s}) = \begin{cases} f(\frac{s'}{s^{nN+1}})s^{nN+1} & \text{if } \; s^{nN+1} > 0 \\ f_\infty(s') & \text{if } \; s^{nN+1} = 0. \end{cases}$$

Let $\mathcal{M}$ be a subclass of probability Radon measures in S_+ which consists of all of such measures as ν in the following: there exist a sequence $\{v_m\} \subset [BV(\Omega)]^N$, a function $v \in [BV(\Omega)]^N$, and a Radon measure λ in $\overline{\Omega} \times \overline{S}_+$ such that $v_m \to v$ strongly in $L^1(\Omega)$, $|\mu_{v_m}| \otimes \delta_{\vec{\mu}_{v_m}(x)} \rightharpoonup \lambda$ in the sense of Radon measures in $\overline{\Omega} \times \overline{S}_+$, and $\nu = \nu_{\lambda,x}$ for one of $x \in \overline{\Omega}$. Now we state assumption (A6):

(A6) $\int_{\overline{S}_+} F(\vec{s})d\nu > F(\int_{\overline{S}_+} \vec{s}d\nu)$ whenever $\nu \in \mathcal{M}$ and $\#\mathrm{spt}\, \nu \geq 2$.

Note that, since f is quasiconvex, the inequality of assumption (A6) always holds with equality.

Let $\overline{u}^{h,(j)}$, u be as in Proposition 3.2. Then there are one parameter families of $\boldsymbol{R}^{nN+1}$-valued Radon measures $\mu_{\overline{u}^{h,(j)}(t,\cdot)}$, $\mu_{u(t,\cdot)}$ in Ω, which are in the sequel simply denoted by μ_t^h, μ_t, respectively. Clearly μ_t^h depends on j, but in the sequel j is fixed and then we do not specify it explicitly. By Proposition 3.2 3) there exists a constant K which is independent of h such that

$$\operatorname*{ess.sup}_{t>0} |\mu_t^h|(\overline{\Omega}) \leq K. \tag{4.2}$$

Then (4.1) and (4.2) imply

$$\operatorname*{ess.\,sup}_{t>0} \Big| \int_{\overline{\Omega}\times\overline{S}_+} \beta(x,\vec{s})\, d|\mu_t^h| \otimes \delta_{\vec{\mu}_t^h(x)} \Big| \le K \sup |\beta| \tag{4.3}$$

for any $\beta \in C^0(\overline{\Omega}\times\overline{S}_+)$. By the use of (4.3) and standard compactness argument we obtain the following lemma (compare to [5, Proposition 4.3]).

Lemma 4.1 *There exists a subsequence of* $\{h\}$ *(still denoted by* $\{h\}$*) and a one parameter family of Radon measures* λ_t *in* $\overline{\Omega}\times\overline{S}_+$, $t \in (0,\infty)$, *such that, for each* $\psi \in L^1(0,\infty)$ *and* $\beta \in C^0(\overline{\Omega}\times\overline{S}_+)$,

$$\lim_{h\to 0}\int_0^\infty \psi(t) \int_{\overline{\Omega}\times\overline{S}_+} \beta(x,\vec{s})\, d|\mu_t^h| \otimes \delta_{\vec{\mu}_t^h(x)} dt = \int_0^\infty \psi(t) \int_{\overline{\Omega}\times\overline{S}_+} \beta(x,\vec{s})\, d\lambda_t dt.$$

The following function

$$\alpha(\vec{s}) := \begin{cases} f_{p^j}(\frac{s'}{s^{nN+1}})s^{nN+1} & \text{if} \quad s^{nN+1} > 0 \\ 0 & \text{if} \quad s^{nN+1} = 0 \end{cases}$$

is continuous in $\overline{S}_+$ by assumptions (A3) and (A4), while Lemma 3.4 3) implies that F is continuous in $\overline{S}_+$. Thus, if we have the following theorem, then we obtain (3.15), (3.18) by Lemma 4.1 with $\beta(x,\vec{s}) = \alpha(\vec{s})\nabla\varphi(x)$, $\beta(x,\vec{s}) = F(\iota_{j,\varepsilon}s', s^{nN+1})$, respectively, and the proof of Theorem 3.3 is complete.

Theorem 4.2 *For* $\mathcal{L}^1$*-a.e.* $t \in (0,\infty)$,

$$\lambda_t = |\mu_t| \otimes \delta_{\vec{\mu}_t(x)}.$$

Before the proof of Theorem 4.2 we sum up properties of λ_t.

Lemma 4.3 *For* $\mathcal{L}^1$*-a.e.* $t \in (0,\infty)$,

1) $\mu_t = |\lambda_t| \llcorner \int_{\overline{S}_+} \vec{s}\, d\nu_{\lambda_t,x}$

2) $|\lambda_t|(A) \ge |\mu_t|(A)$ *for each Borel set* $A \subset \overline{\Omega}$

3) $|\lambda_t|(A) = \int_A D_{|\mu_t|}|\lambda_t|(x)d|\mu_t| + (|\lambda_t| \llcorner Z)(A)$ *for* $A \subset \overline{\Omega}$, *where* $D_{|\mu_t|}|\lambda_t|$ *is the derivative of* $|\lambda_t|$ *with respect to* $|\mu_t|$ *and* Z *is the* $|\mu_t|$*-null set defined by* $Z = \{x; D_{|\mu_t|}|\lambda_t|(x) = \infty\}$

4) $\int_{\overline{S}_+} \vec{s}\, d\nu_{\lambda_t,x} = 0$ *for* $|\lambda_t| \llcorner Z$*-a.e.* x

5) $\operatorname{spt} \nu_{\lambda_t,x} \subset S_0$ *for* $|\lambda_t| \llcorner Z$*-a.e.* x.

Proof. 1) For any $g \in C^0(\overline{\Omega}; \boldsymbol{R}^{nN+1})$ and $\psi \in L^1(0,\infty)$

$$\begin{aligned}
&\int_0^\infty \psi(t) \int_{\overline{\Omega}} g(x)d\mu_t dt = \int_0^\infty \psi(t)[\int_{\overline{\Omega}} g^0(x)dx + \int_{\overline{\Omega}} g'(x)dDu]dt \\
=\ & \lim_{h\to 0}\int_0^\infty \psi(t)[\int_{\overline{\Omega}} g^0(x)dx + \int_{\overline{\Omega}} g'(x)dD\overline{u}^h]dt = \lim_{h\to 0}\int_0^\infty \psi(t)\int_{\overline{\Omega}} g(x)d\mu_t^h dt \\
=\ & \lim_{h\to 0}\int_0^\infty \psi(t)\int_{\overline{\Omega}} g(x)\vec{\mu}_t^h d|\mu_t^h| dt = \lim_{h\to 0}\int_0^\infty \psi(t)\int_{\overline{\Omega}\times\overline{S}_+} g(x)\cdot \vec{s}\, d\lambda_t^h dt \\
=\ & \int_0^\infty \psi(t)\int_{\overline{\Omega}\times\overline{S}_+} g(x)\cdot \vec{s}\, d\lambda_t(x,\vec{s}) dt = \int_0^\infty \psi(t)\int_{\overline{\Omega}} g(x)(\int_{\overline{S}_+} \vec{s}\, d\nu_{\lambda_t,x}) d|\lambda_t| dt,
\end{aligned}$$

where $\lambda_t^h = |\mu_t^h| \otimes \delta_{\vec{\mu}_t^h(x)}$. This shows assertion 1).

2) First we consider the case that A is the intersection of an open set and $\overline{\Omega}$. By assertion 1) we have, for any $g \in C^0(A; \boldsymbol{R}^{nN+1})$,

$$|\int_A g(x) d\mu_t| \leq \int_A |g(x)| d|\lambda_t| \leq \sup |g| |\lambda_t|(A).$$

Taking supremum with respect to $g \in C^0(A; \boldsymbol{R}^{nN+1})$ with $|g| \leq 1$, we obtain $\mu_t(A) \leq |\lambda_t|(A)$.

Let A be any Borel set. For each open set O with $A \subset O$, $\mu_t(A) \leq \mu_t(O \cap \overline{\Omega}) \leq |\lambda_t|(O \cap \overline{\Omega})$. Thus, since $\inf_{A \subset O \cap \overline{\Omega}} |\lambda_t|(O \cap \overline{\Omega}) = |\lambda_t|(A)$, we have $\mu_t(A) \leq |\lambda_t|(A)$.

3) It is the direct consequence of the differentiation theory for Radon measures (see, for example, [11, Theorem 4.7]).

4) By assertions 1) and 3) we have, for any $g(x) \in C^0(\overline{\Omega}; \boldsymbol{R}^{nN+1})$,

$$0 = \int_Z g(x) d\mu_t = \int_Z g(x) \left(\int_{\overline{S}_+} \vec{s}\, d\nu_{\lambda_t,x}\right) d|\lambda_t|.$$

This shows assertion 4).

5) By 4), in particular, we have $\int_{\overline{S}_+} s^{nN+1} d\nu_{\lambda_t,x} = 0$ for $|\lambda_t| \llcorner Z$-a.e. x. Since $s^{nN+1} \geq 0$, we have $s^{nN+1} = 0$ for $\nu_{\lambda_t,x}$-a.e. for $|\lambda_t| \llcorner Z$-a.e. x. Thus assertion 5) holds. Q.E.D.

Lemma 4.4 *For $\mathcal{L}^1$-a.e. $t \in (0,\infty)$, $|\lambda_t \llcorner F|(A) \geq (|\mu_t| \llcorner F(\vec{\mu}_t))(A)$ for each Borel set $A \subset \overline{\Omega}$*

Proof. Lemma 4.3 3) implies, for $\mathcal{L}^1$-a.e. $t \in (0,\infty)$,

$$|\lambda_t| \llcorner (\overline{\Omega} \setminus Z) = |\mu_t| \llcorner D_{|\mu_t|} |\lambda_t|.$$

Hence by Lemma 4.3 1), for $\mathcal{L}^1$-a.e. $t \in (0,\infty)$,

$$\int_{A \setminus Z} d\mu_t = \int_{A \setminus Z} \int_{\overline{S}_+} \vec{s}\, d\nu_{\lambda_t,x} D_{|\mu_t|} |\lambda_t| d\mu_t.$$

For such a t, since Z is a $|\mu_t|$-null set, we have, for $|\mu_t|$-a.e. $x \in \overline{\Omega}$,

$$\vec{\mu}_t(x) = \int_{\overline{S}_+} \vec{s}\, d\nu_{\lambda_t,x} D_{|\mu_t|} |\lambda_t|(x). \tag{4.4}$$

At each point (t,x) such that (4.4) holds, the homogeneity and quasiconvexity of F imply

$$F(\vec{\mu}_t(x)) = F\left(\int_{\overline{S}_+} \vec{s}\, d\nu_{\lambda_t,x} D_{|\mu_t|} |\lambda_t|(x)\right) \leq \int_{\overline{S}_+} F(\vec{s}) d\nu_{\lambda_t,x} D_{|\mu_t|} |\lambda_t|(x). \tag{4.5}$$

By Lemma 4.3 3) again we obtain, for each Borel set $A \subset \overline{\Omega}$,

$$\int_A \int_{\overline{S}_+} F(\vec{s}) d\nu_{\lambda_t,x} D_{|\mu_t|} |\lambda_t|(x) d\mu_t \leq \int_A \int_{\overline{S}_+} F(\vec{s}) d\nu_{\lambda_t,x} d|\lambda_t|. \tag{4.6}$$

Thus the conclusion follows from (4.5) and (4.6). Q.E.D.

Proof of Theorem 4.2. We write $\lambda_t^h = |\mu_t^h| \otimes \delta_{\vec{\mu}_t^h(x)}$. Noting that F is continuous in $\overline{S}_+$, we let $\beta(x, \vec{s}) = F(\vec{s})$ in Lemma 4.1. Then we easily obtain that, for $\mathcal{L}^1$-a.e. $t \in (0, \infty)$,

$$\limsup_{h \to 0} |\lambda_t^h \llcorner F|(\overline{\Omega}) \geq |\lambda_t \llcorner F|(\overline{\Omega}). \tag{4.7}$$

On the other hand (3.12) means

$$\lim_{h \to 0} |\lambda_t^h \llcorner F|(\overline{\Omega}) = (|\mu_t| \llcorner F(\vec{\mu}_t))(\overline{\Omega}) \tag{4.8}$$

for $\mathcal{L}^1$-a.e. $t \in (0, \infty)$. Let t be a number at which all of Lemma 4.4, (4.7), and (4.8) hold. Such a number exists $\mathcal{L}^1$ almost everywhere, and in the sequel we fix it. Then we have $|\lambda_t \llcorner F|(\overline{\Omega}) = (|\mu_t| \llcorner F(\vec{\mu}_t))(\overline{\Omega})$. This and Lemma 4.4 again imply

$$|\lambda_t \llcorner F| = |\mu_t| \llcorner F(\vec{\mu}_t). \tag{4.9}$$

By the definition of $|\lambda_t \llcorner F|$ and $|\mu_t| \llcorner F(\vec{\mu}_t)$, for each Borel set $A \subset \overline{\Omega}$,

$$\int_A F(\vec{\mu}_t(x)) d|\mu_t| = \int_A \int_{\overline{S}_+} F(\vec{s}) d\nu_{\lambda_t, x} d|\lambda_t|. \tag{4.10}$$

In particular, letting $A = Z$, we find $\int_{\overline{S}_+} F(\vec{s}) d\nu_{\lambda_t, x} = 0$ for $|\lambda_t| \llcorner Z$-a.e. x. The definition of F and (1.2) imply $|F(\vec{s})| \geq m|s'|$. Lemma 4.3 5) implies $|s'| \equiv 1$ on spt $\nu_{\lambda_t, x}$ for $|\lambda_t| \llcorner Z$-a.e. x. Thus we have $|\lambda_t|(Z) = \int_Z \int_{\overline{S}_+} d\nu_{\lambda_t, x} d|\lambda_t| \leq m^{-1} \int_Z \int_{\overline{S}_+} F(\vec{s}) d\nu_{\lambda_t, x} d|\lambda_t| = 0$. By Lemma 4.3 4) we conclude

$$|\lambda_t| = |\mu_t| \llcorner D_{|\mu_t|} |\lambda_t|. \tag{4.11}$$

It follows from (4.10) and (4.11) that $F(\vec{\mu}_t(x)) = \int_{\overline{S}_+} F(\vec{s}) d\nu_{\lambda_t, x} D_{|\mu_t|} |\lambda_t|(x)$ for $|\mu_t|$-a.e. $x \in \overline{\Omega}$. Replacing $\vec{\mu}_t(x)$ with the right hand side of (4.4), we obtain, for $|\mu_t|$-a.e. $x \in \overline{\Omega}$,

$$F\left(\int_{\overline{S}_+} \vec{s} d\nu_{\lambda_t, x}\right) = \int_{\overline{S}_+} F(\vec{s}) d\nu_{\lambda_t, x}. \tag{4.12}$$

Since f satisfies (A6) and $\nu_{\lambda_t, x} \in \mathcal{M}$, we have by (4.12) that, for $|\mu_t|$-a.e. $x \in \overline{\Omega}$, spt $\nu_{\lambda_t, x}$ consists of only one point. Let $\vec{s}_x$ be the unique element of spt $\nu_{\lambda_t, x}$. Then (4.4) implies $\vec{\mu}_t(x) = D_{|\mu_t|} |\lambda_t|(x) \vec{s}_x$, which immediately yields $D_{|\mu_t|} |\lambda_t|(x) = 1$ and $\vec{\mu}_t(x) = \vec{s}_x$, for $|\mu_t|$-a.e. $x \in \overline{\Omega}$. By (4.11) we deduce $|\lambda_t| = |\mu_t|$ on $\overline{\Omega}$. Hereby we obtain by Lemma 4.3 2) that, for each $\beta \in C^0(\overline{\Omega} \times \overline{S}_+)$,

$$\int_{\overline{\Omega} \times \overline{S}_+} \beta(x, \vec{s}) d\lambda_t = \int_{\overline{\Omega}} \beta(x, \vec{\mu}_t(x)) d|\mu_t|.$$

This implies the conclusion. Q.E.D.

References

[1] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Science Publication, 2000.

[2] L. C. Evans, *Weak convergence methods for nonlinear partial differential equations*, CBMS, vol. 74, Amer. Math. Soc., 1990.

[3] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, 1992.

[4] D. Fujiwara, A. Inoue, and S. Takakuwa, *A varifold solution of nonlinear wave equation of a membrane*, Proc. Japan Acad. Sci. **60** (1984), 113–116.

[5] D. Fujiwara and S. Takakuwa, *A varifold solution to the nonlinear equation of motion of a vibrating membrane*, Kodai Math. J. **9** (1986), 84–116, correction, ibid. 14 (1991), 310-311.

[6] E. De Giorgi, *New problems on minimizing movements*, Boundary Value Problems for PDE and Applications, Masson, 1993, pp. 81–98.

[7] E. Giusti, *Minimal surfaces and functions of bounded variation*, Birkhäuser, Boston-Basel-Stuttgart, 1984.

[8] A. Hoshiga, *The asymptotic behaviour of the radially symmetric solutions to quasilinear wave equations in two space dimensions*, Hokkaido Math. J. **24** (1995), 575–615.

[9] K. Kikuchi, *An analysis of the nonlinear equation of motion of a vibrating membrane in the space of BV functions*, J. Math. Soc. Japan **52** (2000), 741–766.

[10] ———, *A remark on Dirichlet boundary condition for the nonlinear equation of motion of a vibrating membrane*, Nonlinear Analysis **47** (2001), 1039–1050.

[11] L. Simon, *Lectures on geometric measure theory*, Proceeding of the Centre for Mathematical Analysis, vol. 3, Australian National University, Canberra, 1983.